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LETTER TO THE EDITOR

Non-self-dual solutions of gauge field models in $2n$ dimensions

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Abstract. It is shown that scale-breaking generalized Yang–Mills systems in $2n$ dimensions have hedgehog (instanton) solutions for all $n>1$. For the cases $n=3$ and 4, the equations are solved in the asymptotic regions and these hedgehogs are studied numerically.

The Yang–Mills (YM) model in four dimensions occupies a central place in the theory of fundamental interactions, both in QCD and electroweak dynamics. In addition to the successes of these theories at the perturbative level, there is the important sector of non-perturbative effects in both of them, that must be considered. In the latter area, the central role is played by the instanton field configurations of the given model, which, as solutions to the Euclidean field equations, characterize the vacuum of the theory in question. In the case of pure Yang–Mills theory, these instantons are thoroughly studied.

The generalization of the YM model to higher dimensions, such that the hierarchy of the generalized YM(GYM) models [1] supports instanton solutions, is essentially physically motivated [1, 3]. The motivation is very simply to define models in (even) higher dimensions, which under dimensional reduction yield residual systems that are *also* endowed with instanton solutions. Since some of the components of the connection field in (the higher) $2n$ dimensions will appear as Higgs fields after this descent to $d(<2n)$, this procedure is capable of producing a generalization of the YM-Higgs system which supports a *localized* instanton in d dimensions. The said localization is due to the appearance of an absolute scale in the theory, as a result of the dimensional reduction over a compact coset space.

In particular when $d=4$, these supply various candidates to generalize both QCD and electroweak dynamics, which are endowed with *localized* instantons, central to the study of non-perturbative effects in both these theories. The advantage of this localization property in the case of (generalized) QCD is the removal of the well known difficulties posed by the arbitrary scale of the YM instantons, while the occurrence of a Higgs multiplet in a (generalized) electroweak model is a self-evident requirement.

According to the above prescription for constructing localized instantons in four dimensions, the first step is the generalization of the YM system to *all* even dimensions, such that the GYM models support instanton solutions. For the above prescription to work, namely that the residual system support instantons, the higher dimensional model in the corresponding Euclidean dimensions must support instantons. While explicit solutions [3, 4] are found in $d=4p$, the corresponding task in $d=2(2p+1)$ has not been

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performed, even though we believe such solutions to the appropriate GYM models exist. The reason for this omission was that the latter class of solutions cannot be found explicitly and must be found numerically. It is the main aim of this work to supply these solutions in $2(2p+1)$ dimensions, completing the demonstration that GYM systems in all even dimensions support instanton solutions. In addition, we shall draw some general conclusions on the infinite asymptotic behaviour of instantons in gauge field systems, which we note at the end of this letter.

The study of instanton solutions of the generalized Yang–Mills (GYM) systems [1] defined on even-dimensional Euclidean spaces, has thus far been restricted to the case where the solutions in question satisfy the first-order self-duality equations. These pertain to the conformally invariant SO($4p$) GYM systems [2] defined in $4p$ dimensions, whose action in a topological sector is minimized globally by the self-duality condition

$$F(2p) = {}^*F(2p). \tag{1}$$

Here, $F(2p)$ is the p-fold totally antisymmetrized product of the curvature 2-form $F(2)$ and $^*F(2p)$ is its Hodge dual. Equations (1) have non-trivial chiral SO($4p$) instanton solutions [3–5] on R_{4p}, which are the absolute minima of the action corresponding to the Lagrangian

$$L(p, p) = \mathrm{tr}\, F(2p)^2. \tag{2}$$

There is another, more general type of self-duality equation in $2n = 2(q+p)$ dimensions, namely

$$\kappa^{2(q-p)} F(2p) = {}^*(F(2q))(2p) \tag{3}$$

with $q > p$ and in which a constant κ with the dimension of length plays a role. Equation (3) solves the Euler–Lagrange equations pertaining to the Lagrangian

$$L(q, p) = \mathrm{tr}\, [\kappa^{4(q-p)} F(2p)^2 + F(2q)^2]. \tag{4}$$

The only known non-trivial solutions to the self duality equations (3) are those on compact symmetric spaces. Such solutions were found in [6] for arbitrary values of (q, p) for S^{2n} and CP^n, and for the cases with $q = 1$, for S^{2n}, CP^n and HP^n in [7]. They have only trivial, pure gauge, solutions on $R_{4(p+q)}$. The reason for this is that the curvature field strength of an instanton decays as a power of the radius at infinity.

The purpose of the present note is to present the spherically symmetric, non-minimal instanton solutions to the Euler–Lagrange equations of Lagrangians of type (4). Such instantons were mentioned in [1] but have hitherto not been studied. These non-minimal instantons in (Higgs-free) gauge field theory are the analogues of the Skyrme [8] hedgehog in the extended $O(4)$ sigma-model. Both are solutions of the Euler–Lagrange equations but *not* the respective self-duality equations, for the same reason. Like the Skyrme model, the Lagrangians (4) break scale invariance by virtue of the presence of κ, which plays the role of supplying an absolute scale with respect to which the instanton is localized. This is also the case with the Skyrmion.

That topologically stable solutions to the equations of (4) exist, follows from the inequality

$$L(q, p) \geqslant 2\kappa^{2(q-p)}\, \mathrm{tr}\, F(2p)^*(F(2q))(2p) \tag{5}$$

where the right-hand side is evidently proportional to the totally antisymmetrized $n = (p+q)$-fold product $\mathrm{tr}\, F \wedge F \wedge \ldots \wedge F$ which is the nth Chern–Pontryagin(C-P) density. It follows that the integral of the density (4), the action, is bounded from below by a surface integral which takes a non-zero (finite) value provided that suitable boundary conditions are satisfied. The latter require that the curvature field strength decays at infinity with some power of the radius r, which is inconsistent with the constraints of the self-duality equations (3).

Before proceeding to a detailed study of these solutions, we point out that there is a larger class of Lagrangians than the class given by (4) whose equations of motion have non-minimal hedgehog solutions. This is seen directly by noting that the topological inequality (5) is not invalidated when a positive definite quantity is added to its left-hand side. There is in fact a unique set of such terms that can be employed in any dimension $2n$. These additional terms must clearly be 'Lorentz' and gauge invariant, and we require further that they satisfy the 'causality' property that there occurs no higher power than the second power of the gradient of any given component of the connection gauge field A_μ. These requirements narrow down the choice for such terms to

$$L^{(m)} = \mathrm{tr}\, F(2m)^2 \tag{6}$$

where $m \leqslant n$, and there are $n-2$ such terms. Thus we may give the most general Lagrangian in any given dimension $2n$ as

$$L = L(q,p) + \tilde{L}_1 = L(q,p) + \sum_{1}^{n-2} \lambda^{(m)} L^{(m)} \tag{7}$$

which is defined in terms of (4) and (6). We shall refer to the class (4) as *minimal* Lagrangians, referring to the fact that these are the ones that are necessary to satisfy the topological inequality (5). For $n>4$, there is more than one choice for a minimal Lagrangian, corresponding to the multiple choices for partitions q and p of n, such that $q \neq p$.

We shall now test the above arguments in favour of the existence of non-minimal GYM hedgehogs in $2n$ dimensions by considering the $n=3$ and 4 cases in detail. In each case, we shall examine both the minimal Lagrangians (4) and one example in each case of an extended Lagrangian of the form (7). We shall restrict our attention to spherically symmetric, hedgehog field configurations, so that the variational problem under consideration will be reduced to a one-dimensional one.

The only remaining choice is that of the gauge group. The solutions we shall study pertain to the case where the gauge field model given by (4) or (7) on $R_{2(p+q)}$, namely in $2n$ Euclidean dimensions, is defined for the gauge field connection taking its values in $SO_{\pm}(2n)$, chiral SO($2n$). Here we are guided to this choice by our desire for imposing spherical symmetry on the gauge field in a simple and natural way. Chiral SO($4p$), with $n=2p$, also happens to be the gauge group of the scale-invariant GYM models on R_{4p} for which both spherically symmetric [3] and axially symmetric [5] solutions are known. In the latter case, where the self-duality equations are overdetermined [9], this choice of gauge group can also be said to be made for the reason that strict symmetries must be imposed to compensate for the overdetermination, *except* in the case of $p=1$ or the usual YM model for which instanton [10] solutions are not restricted [11] to be at least axially symmetric.

Following our adaptation [3] of the Schwarts [12] formalism for spherically symmetric gauge fields, we express the components of the curvature field strength

$F_{\mu\nu}=(F_{ij}, F_{i,2n})$, $i, j=1, \ldots, 2n-1$, of the chiral SO($2n$) gauge field at the north pole of the sphere S^{2n-1} in R_{2n} as

$$F_{ij}=-\frac{1}{r^2}(1-f^2(r))\Sigma_{ij} \tag{8a}$$

$$F_{id}=-\frac{1}{r}f'(r)\Sigma_{id} \tag{8b}$$

where $d=2n$, r is the radial coordinate, and $\Sigma_{\mu\nu}=(\Sigma_{ij}, \Sigma_{id})$ denote the generators of SO(d) of either chirality. In this note, we shall consider in detail the two cases $d=6$ and 8 respectively, which is sufficient to explain how to construct such hedgehogs in all even dimensions.

In the following, we shall employ the dimensionless variable $\rho=\kappa^{-1}r$ instead of the radial coordinate, where κ is the dimensional constant appearing in (3)–(5). Thus, as in the Skyrme model [13], the localization of the hedgehog due to a falloff with a power of ρ will be controlled by the size of the dimensional constant κ.

Six dimensions: Here, there is only one choice for the minimal Lagrangian, namely that with $p=1$ and $q=2$ in (4). After the imposition of spherical symmetry using (8a, 8b), the residual one-dimensional Lagrangian which is proportional to $r^5L(q=2, p=1)$ can be expressed in terms of ρ as

$$L_1(2, 1)=\kappa\left\{\rho^3\left[f_\rho^2+\frac{2}{\rho^2}(1-f^2)^2\right]+\frac{1}{\rho}(1-f^2)^2\left[2f_\rho^2+\frac{1}{\rho^2}(1-f^2)^2\right]\right\}. \tag{9}$$

Apart from (9), the most general Lagrangian of the form (7) in six dimensions is also unique, since there is only one possible candidate for the additional term of the type (6). This term is the one given by $m=3$ in (6)

$$\tilde{L}=\frac{\eta^8}{\kappa^7\rho^5}(1-f^2)^4f_\rho^2=\kappa\left\{\frac{\lambda}{\rho^5}(1-f^2)^4f_\rho^2\right\} \tag{10}$$

where η is another constant with the dimension of a length and $\lambda=(\eta/\kappa)^8$ is a dimensionless coupling strength. Thus the most general Lagrangian in six dimensions for which we can expect a non-trivial hedgehog is

$$L_6=L_1(2, 1)+\tilde{L}_1. \tag{11}$$

The requirement that the one-dimensional integrals of (9), (10) and (11) converge, dictates that each (positive definite) term in the integrand vanishes asymptotically. Also, the solution must not be singular at $\rho=0$. The result of these requirements is

$$-1\xleftarrow[\rho\to 0]{}f(\rho)\xrightarrow[\rho\to\infty]{}+1. \tag{12}$$

Evaluating the surface integral corresponding to the C-P density given on the right-hand side of (5) with the infinite limit of (12), yields unit topological charge. This is the charge of the hedgehog.

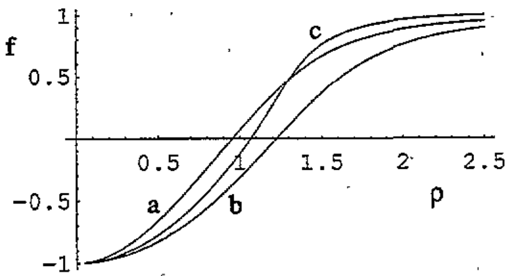

Figure 1. f as a function of ρ for the cases: (a) six dimensions with $\lambda=0$; (b) six dimensions with $\lambda=2$; (c) eight dimensions with $\lambda_1=\lambda_2=0$.

The Euler–Lagrange equations are

$$\left[\rho^3+\frac{2}{\rho}(1-f^2)^2+\frac{\lambda}{\rho^5}(1-f^2)^4\right]f_{\rho\rho}$$
$$+\left[3\rho^2-\frac{2}{\rho^2}(1-f^2)^2-\frac{4}{\rho}(1-f^2)ff_\rho-\frac{5\lambda}{\rho^6}(1-f^2)^4-\frac{4\lambda}{\rho^5}(1-f^2)^3ff_\rho\right]f_\rho$$
$$+4\rho(1-f^2)f+\frac{4}{\rho^3}(1-f^2)^3f=0. \tag{13}$$

The equation of motion corresponding to the minimal Lagrangian (9) can be read off (13) by setting $\lambda=0$, and otherwise this is the equation of motion for the full system (11).

Setting $f=1+F$ with small F in the (infinite) asymptotic region and retaining only the leading terms in (13), the latter reduces to

$$\rho^2F_{\rho\rho}+3\rho F_\rho-8F=0 \tag{14}$$

whose solution, in terms of an arbitrary constant A, is

$$f(\rho)\xrightarrow[\rho\to\infty]{}1+\frac{A}{\rho^4}+o(\rho^{-4}) \tag{15}$$

which is a power fall-off as in the case of the Skyrmion [13]. In the other asymptotic region near the origin, we find the following solution

$$f(\rho)\xrightarrow[\rho\to 0]{}-1+C\rho^2-\frac{8C^4+3C^2-16\lambda C^6}{4(1+8C^2+16\lambda C^4)}\rho^4+o(\rho^4) \tag{16}$$

where C is an arbitrary constant. Both solutions (15) and (16) hold whether or not $\lambda=0$. Using this solution at the origin we have numerically integrated (13) for different values of the parameter λ. In figures 1(a) and 1(b), the solutions are plotted for $\lambda=0$, where $C=1.49$, and $\lambda=2$, where $C=0.69$, respectively.

Eight dimensions: Here, there are two possible partitions of $n=q+p$ but only one of them, $p=1$ and $q=3$, is relevant to us since the other one, $p=q=2$, is the case where the self-duality equations have solutions. As the latter are absolute minima, we cannot expect to find any non-self-dual solutions. We therefore restrict our attention to the

($p=1, q=3$) case for the minimal Lagrangian (4). There are, however, different choices for the additional Lagrangian terms (6), namely those with $m=2$ and 4, or both.

The minimal Lagrangian for the one-dimensional subsystem obtained after the imposition of spherical symmetry, namely using the formulas (8*a*, 8*b*), is

$$L_1(3,1)=\kappa^3\left\{\rho^5\left[f_\rho^2+\frac{3}{\rho^2}(1-f^2)^2\right]+\frac{2}{\rho^3}(1-f^2)^4\left[3f_\rho^2+\frac{1}{\rho^2}(1-f^2)^2\right]\right\}. \tag{17}$$

This is the analogue of (9), and is proportional to $r^7L(q=3,p=1)$. The additional Lagrangian terms analogous to (10) are those with $m=2$ and 4 mentioned above

$$\begin{aligned}\tilde{L}_1&=\frac{\eta_1^4}{\kappa}\rho(1-f^2)^2\left[f_\rho^2+\frac{1}{\rho^2}(1-f^2)^2\right]+\frac{\eta_2^{12}}{\kappa^9\rho^7}(1-f^2)^6f_\rho^2\\&=\kappa^3\left\{\lambda_1\rho(1-f^23^2\left[f_\rho^2+\frac{1}{\rho^2}(1-f^2)^2\right]+\frac{\lambda_2}{\rho^7}(1-f^2)^6f_\rho^2\right\}\end{aligned} \tag{18}$$

where $\lambda_1=(\eta_1/\kappa)^4$ and $\lambda_2=(\eta_2/\kappa)^{12}$ are two dimensionless coupling strengths. The most general Lagrangian in eight dimensions with hedgehog solutions is then

$$L_8=L_1(3,1)+\tilde{L}_1. \tag{19}$$

The requirements of finite action and regularity result in the same asymptotic conditions as in the six-dimensional example, namely (12). We expect that these will be the appropriate asymptotic conditions for GYM hedgehogs in all $2n$ dimensions, and that in each case this will lead to unit topological charge given by the integral of the right-hand side of (5), the nth C–P density. The Euler–Lagrange equations are

$$\begin{aligned}&\left[\rho^5+\frac{6}{\rho^3}(1-f^2)^4+\lambda_1\rho(1-f^2)^2+\frac{\lambda_2}{\rho^7}(1-f^2)^6\right]f_{\rho\rho}\\&\quad+\left[5\rho^4-\frac{18}{\rho^4}(1-f^2)^4-\frac{24}{\rho^3}(1-f^2)^3ff_\rho+\lambda_1(1-f^2)^2\right.\\&\quad\left.-2\lambda_1\rho(1-f^2)ff_\rho-\frac{7\lambda_2}{\rho^8}(1-f^2)^6-\frac{6\lambda_2}{\rho^7}(1-f^2)^5ff_\rho\right]f_\rho\\&\quad+6\rho^3(1-f^2)f+\frac{12}{\rho^5}(1-f^2)^5f+\frac{4\lambda_1}{\rho}(1-f^2)^3f=0\end{aligned} \tag{20}$$

and as before, setting $\lambda_1=\lambda_2=0$, we have the equations of motion for the minimal model.

Again, expressing $f(\rho)=1+F(\rho)$ at infinity, and retaining only the leading terms in (20), we find

$$\rho^2F_{\rho\rho}+5\rho F_\rho-12F=0 \tag{21}$$

whose solution, in terms of an arbitrary constant A, is

$$f(\rho)\xrightarrow[\rho\to\infty]{}1+\frac{A}{\rho^6}+o(\rho^{-6}).$$

It can be seen from (15) and (22) that the power fall-off of the GYM hedgehog in $2n$ dimensions is $2(n-1)$. The other asymptotic solution, near the origin, is found to be

$$f(\rho) \xrightarrow[\rho\to 0]{} -1 + C\rho^2 - \frac{96C^6 + 9C^2 + 20\lambda_1 C^4 - 192\lambda_2 C^8}{10 + 960C^4 + 40\lambda_1 C^2 + 640\lambda_2 C^6}\rho^4 + o(\rho^4) \tag{22}$$

where C is an arbitrary constant which will be fixed in the numerical integration of the full equation (20).

The numerical integration of (20) has been performed for different choices of λ_1 and λ_2. For $\lambda_1 = \lambda_2 = 0$, where $C = 0.92$, the solution is plotted in figure 1(c).

The numerical analysis, illustrated by figure 1(a), 1(b) and 1(c) shows that our models provide us with families of non-self-dual solutions with a range of quantitatively different features. This completes the main task of the present work, namely that of showing that instanton solutions are supported by the appropriate GYM models in all $2(2p+1)$ dimensions in addition to the previously known cases in all $4p$ dimensions, i.e. this covers all even dimensions $2n$.

We finish this letter by noting a general asymptotic property of instanton solutions in gauge theories in which the gauge field does not interact with a Higgs field. It follows from the infinite asymptotic solutions (15) and (22), that the connection fields $A_\mu = (A_i, A_d)$, $i = 1, \ldots, 2n-1$, at the north pole

$$A_i = f(r)\Sigma_{id}, \quad A_d = 0 \tag{23}$$

corresponding to the field strengths (8), are *asymptotically pure gauge*. This can most conveniently be demonstrated by expressing the connection field in a covariant gauge

$$\tilde{A}_\mu = \frac{2}{r^2}(1 - k(r))\Sigma_{\mu\nu}x_\nu \xrightarrow[r\to\infty]{} \omega^{-1}\partial_\mu \omega \tag{24}$$

where the two radial functions are related as $f(r) = 1 - 2k(r)$, and

$$\omega = \frac{1}{r}\Sigma_\mu^{\pm} x_\mu \qquad \Sigma_\mu^{\pm} = \frac{1}{2}(1 \pm \Gamma_{2n+1})\Gamma_\mu \tag{25}$$

In (25), Γ_μ are the gamma matrices in $d = 2n$ dimensions and Γ_{d+1} is the corresponding chiral matrix. The asymptotic property (24) holds for all the instantons of the GYM models in $d = 2n$ dimensions, since we know it to be the case for the scale invariant ones [2] already. It should be noted, however, that only in the $n = 2$, the usual four dimensional YM case, is it that the matrix function ω happens to be an element of the gauge group, chiral SO($2n$). In that case, the group happens to coincide with SU(2), which has $3 = 2n - 1$ parameters. In all other cases with $n > 2$, the number of parameters in (25), $2n - 1$, is increasingly smaller than the number of parameters in the gauge group SO($2n$).

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